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Abstract. In this paper I construct lattice models with an underlying $U_q osp(2,2)$ superalgebra symmetry. I find new solutions to the graded Yang–Baxter equation. These *trigonometric* R-matrices depend on *three* continuous parameters: the spectral parameter, the deformation parameter q and the $U(1)$ parameter, b, of the superalgebra. It must be emphasized that the parameter q is generic and the parameter b does not correspond to the 'nilpotency' parameter of [15]. The rational limits are given; they also depend on the $U(1)$ parameter and this dependence cannot be rescaled away. I give the Bethe ansatz solution of the lattice models built from some of these R-matrices, while for other matrices, due to the particular nature of the representation theory of $osp(2,2)$, I conjecture the result. The parameter b appears as a continuous generalized spin. Finally I briefly discuss the problem of finding the ground state of these models.

1. Introduction

Lattice models provide us with one way to regularize field theories by cutting off the short distance divergences. Lattices models also lead to physical models of statistical and condensed matter physics (like the Ising model, the Hubbard model, polymer models, etc). Two-dimensional $N=2$ superconformal field theories, which appear in the study of string theories, are at least as appealing as lattice models. It is the existence of a topological sector, for which a semiclassical approach gives exact quantum results, that renders $N=2$ theories so attractive. In this sense $N=2$ theories are simpler than non-supersymmetric theories. By studying lattice analogues of $N=2$ models we hope to recover the 'simplified' structure of the field theory at the lattice level and to apply $N=2$ supersymmetry to physical models through the lattice description.

A whole family of lattice analogues of $N=2$ coset models was constructed in [1]. The presence of a topological sector was used to obtain these models. However, the supersymmetry does not seem to be realized on the lattice. In an attempt to build lattice models through a direct approach and with some degree of supersymmetry on the lattice, it is natural to consider bosonic and fermionic lattice variables. In recent years quantum groups have emerged as a common underlying symmetry of field theories and lattice models [2]. For instance, $U_{q_+} su(2) \otimes U_{q_-} osp(2,2)$ was found to be an underlying symmetry of $N=2$ superconformal theories, for the holomorphic part, in the Neveu–Schwarz (NS) sector [3].

Let $U_q\hat{\mathcal{G}}$ be the affine q-deformed universal enveloping algebra for a Lie algebra $\mathcal{G}$. A general method for obtaining lattice models, with states belonging to representations of

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the Lie algebra, and which have an underlying $U_q\hat{\mathcal{G}}$ symmetry, is by making the transfer matrix out of products of R-matrices. The corresponding lattice models are called vertex models. Such a method generalizes naturally to superalgebras. In what follows I consider the $osp(2,2)$ superalgebra. This is a subalgebra of the $N=2$ superconformal algebra. In looking for supersymmetry on the lattice, it is this subalgebra that one hopes to realize on the lattice.

In this paper I construct lattice models built from $U_q osp(2,2)$ R-matrices. I first obtain new trigonometric solutions for the Yang–Baxter equation. These solutions depend on an additional continuous parameter b for generic values of q. The rational forms of these matrices also depend on b. (Trigonometric solutions were constructed in [4] for the *fundamental* representations of simple Lie superalgebras.) I then perform an algebraic Bethe ansatz to diagonalize the transfer matrix with periodic boundary conditions. The parameter b appears as a generalized spin in the Bethe ansatz equations. I briefly discuss the problem of determining the ground state and the dependence of the central charge and conformal weights on the the parameter b.

2. The superalgebra $osp(2,2)$

2.1. Definitions and notation

The $osp(2,2)$ superalgebra is eight-dimensional and has rank two. There are four even generators, $S_\pm$, S_3 and B, and four odd generators, $V_\pm$ and $\overline{V}_\pm$. Let $[\,,\,]$ denote the usual commutator, and $\{\,,\,\}$ denote the anti-commutator. The commutation relations are given by

$$[S_3, S_\pm] = \pm S_\pm \qquad [S_+, S_-] = 2S_3 \tag{1}$$

$$[P_\pm, V_\pm] = 0 \qquad [P_\mp, V_\pm] = \pm V_\pm \qquad [P_\pm, \overline{V}_\pm] = \pm\overline{V}_\pm \qquad [P_\mp, \overline{V}_\pm] = 0 \tag{2}$$

$$\{V_i, V_j\} = \{\overline{V}_i, \overline{V}_j\} = 0 \qquad i, j = \pm \tag{3}$$

$$\{V_+, \overline{V}_-\} = -\tfrac{1}{2}P_+ \qquad \{\overline{V}_+, V_-\} = -\tfrac{1}{2}P_- \tag{4}$$

$$\{V_\pm, \overline{V}_\pm\} = \pm\tfrac{1}{2}S_\pm \tag{5}$$

where, following the notation of [5],

$$P_\pm = S_3 \mp B. \tag{6}$$

The generators also satisfy the graded Jacobi identity.

The even sub-algebra is $su(2)\oplus u(1)$, and is generated by $S_\pm$, S_3 and B or, equivalently, $S_\pm$ and $P_\pm$. The generators $V_\pm$ (resp. $\overline{V}_\pm$) form $su(2)$ spin-$\frac{1}{2}$ tensors (spinors) with 'hypercharge' $B = \frac{1}{2}$ (resp. $B = -\frac{1}{2}$).

The $osp(2,2)$ superalgebra is a subalgebra of the $N=2$ superconformal algebra in the NS sector. Using the notation of [6] one has

$$L_0 = -S_3 \qquad L_\pm = \pm S_\pm \qquad J_0 = 2B \qquad G^+_{\pm\frac{1}{2}} = 2\overline{V}_\pm \qquad G^-_{\pm\frac{1}{2}} = 2V_\pm. \tag{7}$$

2.2. Some $osp(2,2)$ representations

Unlike ordinary Lie algebras, there are two types of representation for most superalgebras. The *typical* representations are irreducible and are similar to the usual representations of ordinary Lie algebras. The values of the Casimirs, the central elements, for a given typical representation, are unique to the representation. The *atypical* representations have no counterpart in the ordinary Lie algebra representations. They can be irreducible or not fully reducible (read reducible but indecomposable). The Casimirs for two atypical representations can take the same values.

The superalgebra $osp(2,2)$ is isomorphic to the superalgebra $spl(2,1)$. The representation theory of the latter superalgebra was studied in [7, 8]. Generically a representation (b, s) ($b \in \mathbb{C}$, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$) contains four $su(2) \oplus u(1)$ multiplets:

$$|b, s, s_3\rangle \qquad s_3 = -s, -s+1, \ldots, s-1, s \qquad \text{if } s \geqslant 0 \tag{8}$$

$$|b + \tfrac{1}{2}, s - \tfrac{1}{2}, s_3\rangle \qquad s_3 = -s + \tfrac{1}{2}, \ldots, s - \tfrac{3}{2}, s - \tfrac{1}{2} \qquad \text{if } s \geqslant \tfrac{1}{2} \tag{9}$$

$$|b - \tfrac{1}{2}, s - \tfrac{1}{2}, s_3\rangle \qquad s_3 = -s + \tfrac{1}{2}, \ldots, s - \tfrac{3}{2}, s - \tfrac{1}{2} \qquad \text{if } s \geqslant \tfrac{1}{2} \tag{10}$$

$$|b, s - 1, s_3\rangle \qquad s_3 = -s + 1, \ldots, s - 2, s - 1 \qquad \text{if } s \geqslant 1. \tag{11}$$

The action of the four even generators on these multiplets is the one implied by the notation. The four odd generators, in contrast, interpolate between different multiplets. The precise action of these generators, which can be inferred from the defining equations (1)–(5), can be found in [7, 8]. I shall give later the complete matrix form, of all the generators, for the particular representations I consider.

If $b \neq \pm s$ the representation is denoted by $[b, s]$ and is typical; the quadratic and cubic Casimirs do not vanish. The representation $[b, s]$ has dimension $8s$.

When $b = \pm s$ several kinds of atypical representations arise. Both Casimirs vanish, and yet these representations are not the trivial one-dimensional null representation. One kind has dimension $4s + 1$ and is denoted by $[s]_\pm$. To obtain $[s]_+$ (resp. $[s]_-$) one drops the two multiplets $(b - \frac{1}{2}, s - \frac{1}{2})$ and $(b, s - 1)$ (resp. $(b + \frac{1}{2}, s - \frac{1}{2})$ and $(b, s - 1)$). These representations are irreducible. Other kinds of atypical representation, for which all multiplets are kept, have dimension $8s$. They contain two representations of the previous type with one representation being an invariant subspace of the whole representation. They are therefore not fully reducible.

Other types of atypical representations with different dimensionalities exist.

3. $U_q osp(2, 2)$

I show how the universal enveloping algebra $U osp(2, 2)$ is deformed. The coproduct is introduced in order to define the universal, spectral parameter-independent R-matrix, which depends on q. This matrix will be useful in obtaining spectral parameter-dependent R-matrices, $R(x, q)$, which give integrable lattice models. The matrix elements of the matrix $R(x, q)$ give the Boltzmann weights of an integrable lattice model. The coproduct will also be used to obtain another spectral parameter-dependent R-matrix, which is one ingredient in the Bethe ansatz diagonalization.

3.1. *The q-deformed relations*

I consider the q-deformation $U_q osp(2, 2)$ of the universal enveloping algebra, $U osp(2, 2)$, obtained in [5]. Another deformation exists in [9]. The latter deformation relies on a harmonic oscillator representation of the superalgebra.

The q-deformation considered here is precisely the one that appears as an underlying quantum group symmetry of $N = 2$ superconformal theories in the NS sector [3]. There is also another reason for this choice. Superalgebras, unlike ordinary Lie algebras, admit, in general, more than one inequivalent basis of simple roots. It turns out that one can construct $N = 2$ supersymmetric Toda field theories only if one chooses a purely fermionic simple root system [10]. This should be taken as an additional hint if one has in mind the construction of a supersymmetric theory. The foregoing deformation is based on the choice of a purely fermionic simple root system.

The universal enveloping algebra $Uosp(2,2)$ is generated by $P_\pm, V_\pm, \overline{V}_\pm$. This corresponds to an implicit choice of two purely fermionic simple roots for the basis. The relations (2), (3) and (5) are kept unchanged except for the fact that the generators are the q-deformed ones. The two relations (4) are replaced by

$$\{2V_+, 2\overline{V}_-\} = [-2P_+] \qquad \{2\overline{V}_+, 2V_-\} = [-2P_-] \tag{12}$$

where

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{13}$$

The q-deformations of relations (1) are obtained from (5) and (12). The q-generators $S_\pm, S_z$ do *not* satisfy the usual q-deformed $su(2)$ relations:

$$[S_3, S_\pm] = \pm S_\pm \qquad [S_+, S_-] = [2S_3]. \tag{14}$$

It is easy to verify that the first relation in (1) remains the same while the second relation becomes

$$\begin{aligned}[S_+, S_-] = &-V_-\overline{V}_+(q^{2P_++1} + q^{-2P_+-1}) - \overline{V}_+V_-(q^{2P_+-1} + q^{-2P_++1}) \\ &-\overline{V}_-V_+(q^{2P_-+1} + q^{-2P_--1}) - V_+\overline{V}_-(q^{2P_--1} + q^{-2P_-+1}).\end{aligned} \tag{15}$$

This relation collapses to the usual one, $[S_+, S_-] = [2S_3]$, for certain representations. It is interesting to note that the $su(2)$ subalgebra is not deformed to a $U_qsu(2)$ subalgebra of $U_qosp(2,2)$. This seems to be the case for superalgebras in general (see [11] for example).

3.2. Coproducts

The usual tensor product construction, *for the operators*, of the algebra is not compatible with the q-deformed commutation relations of the (Hopf) superalgebra $U_qosp(2,2)$. However, a q-deformed tensor product can be defined. This new tensor product is conveniently encoded in the coproduct Δ. The defining relations are given by [5]

$$\Delta(P_\pm) = P_\pm \otimes 1 + 1 \otimes P_\pm \tag{16}$$

$$\Delta(V_\pm) = q^{P_\pm} \otimes V_\pm + V_\pm \otimes q^{-P_\pm} \qquad \Delta(\overline{V}_\pm) = q^{P_\mp} \otimes \overline{V}_\pm + \overline{V}_\pm \otimes q^{-P_\mp}. \tag{17}$$

Again $\Delta(S_\pm)$ are not given by the usual $U_qsu(2)$ expressions; they can be obtained using (5) and (17).

Throughout this section the explicit tensor product sign $\otimes$ is graded. This means that the following rule is applied: minus signs are generated each time two odd elements, generators and/or vectors (in some representation) are 'commuted' through one another.

There is also another coproduct, $\overline{\Delta}$, with defining relations obtained by pairwise permuting the generators in the defining relations of Δ (i.e. $A \otimes B \to B \otimes A$). Equivalently, one has $\overline{\Delta}_q = \Delta_{q^{-1}}$, for the explicit dependence on q in the defining relations (17). However, as happens with q-deformations of ordinary Lie algebras, it is possible to show that there exists a 'matrix' R [5], in $U_qosp(2,2) \otimes U_qosp(2,2)$, such that the coproducts Δ and $\overline{\Delta}$ are related by

$$\overline{\Delta}(g)R = R\Delta(g) \tag{18}$$

for all elements g of $U_qosp(2,2)$. The 'matrix' R is universal because it is representation independent and depends only on the algebra.

The matrix R satisfies the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{19}$$

where, for $R = \sum a \otimes b$, $R_{12} = \sum a \otimes b \otimes 1$, $R_{13} = \sum a \otimes 1 \otimes b$, etc.

It is the extension of this construction to affine (Kac–Moody) q-deformed algebras in general which gives the R-matrix an additional dependence on the spectral parameter x.

4. A new R-matrix

I now determine new R-matrices for a set of four-dimensional representations of $osp(2,2)$. The fundamental representation, $[0, \frac{1}{2}]$, of the supergroup $OSp(2,2)$ is contained in the set of four-dimensional representations $[b, \frac{1}{2}]$.

I first define and then q-deform the representations $[b, \frac{1}{2}]$. I obtain the matrix $R(q,b)$ and its spectral decomposition, and then find the 'affinized', i.e. spectral parameter dependent, version of the matrix $R(q,b)$.

4.1. *The* $[b, \frac{1}{2}]$ *representations*

I consider the four-dimensional typical representations $[b, \frac{1}{2}]$, in the notation of section 2.2, where, *a priori*, $b \in \mathbb{C} - \{\pm\frac{1}{2}\}$. One has inequivalent representations for different values of b. From (8)–(11), one finds four vectors which I choose as a basis:

$$b_1 = |b, \tfrac{1}{2}\rangle \qquad f_1 = |b - \tfrac{1}{2}, 0\rangle \qquad f_2 = |b + \tfrac{1}{2}, 0\rangle \qquad b_2 = |b, -\tfrac{1}{2}\rangle \tag{20}$$

where b_1 and b_2 are even (bosonic) and f_1 and f_2 are odd (fermionic). The *relative* parities, two even and two odd, of those vectors can be chosen without loss of generality. The two resulting R-matrices are related by a similarity (gauge) transformation.

The generators for the representation $[b, \frac{1}{2}]$ are given by [7, 8]

$$V_+ = \begin{pmatrix} 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \overline{V}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \end{pmatrix} \tag{21}$$

$$\overline{V}_+ = \begin{pmatrix} 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad V_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \end{pmatrix} \tag{22}$$

$$P_+ = \begin{pmatrix} \frac{1}{2} - b & 0 & 0 & 0 \\ 0 & \frac{1}{2} - b & 0 & 0 \\ 0 & 0 & -\frac{1}{2} - b & 0 \\ 0 & 0 & 0 & -\frac{1}{2} - b \end{pmatrix}$$

$$P_- = \begin{pmatrix} b + \frac{1}{2} & 0 & 0 & 0 \\ 0 & b - \frac{1}{2} & 0 & 0 \\ 0 & 0 & b + \frac{1}{2} & 0 \\ 0 & 0 & 0 & b - \frac{1}{2} \end{pmatrix}.$$

The four parameters appearing in (21) and (22) are constrained by

$$4\alpha\gamma = 1 + 2b \qquad 4\beta\epsilon = 1 - 2b. \tag{23}$$

Thus there are two free parameters which correspond to arbitrary relative normalizations of the $su(2)$ doublet (b_1, b_2), and the two singlets f_1 and f_2.

For $b \neq \frac{1}{2}$ these representations are typical while for $b = \pm\frac{1}{2}$ they are atypical. Note that b can take complex values in what follows if one does not consider Hermitian representations.

The q-deformed version of this representation is obtained as follows. The generators $P_\pm$ remain undeformed. The matrix form of the four odd generators remains unchanged. However, the four parameters satisfy q-deformed constraints

$$4\alpha\gamma = [1+2b] \qquad 4\beta\epsilon = [1-2b] \tag{24}$$

where $[x]$ was defined in (13). If q goes to one then equations (24) yield equations (23) and one recovers the undeformed representation.

4.2. The matrix $R(q, b)$

The matrix $R(q, b)$ is obtained by plugging the four-dimensional matrices, (21)–(22) and $P_\pm$, subject to the deformation constraints (24), into the universal R-matrix found in [5]. I shall give the result in a slightly different form below.

I now put the R-matrix just obtained in a form which is convenient to give it a spectral parameter dependence. Recall that $R(q, b)$ acts in $V \otimes V$. Because the two representations on the left and the right of this tensor product are equal, one can define a, graded, permutation operator, $\mathcal{P}$. It satisfies

$$\mathcal{P}|u\rangle \otimes |v\rangle = (-1)^{\varepsilon_u \varepsilon_v}|v\rangle \otimes |u\rangle \tag{25}$$

for two vectors, u and v, of definite parity, ε_u and ε_v, in the representation space V. One can then define the matrix $\check{R}$ given by

$$\check{R} = \mathcal{P}R. \tag{26}$$

From (18) one gets

$$\check{R}\Delta(g) = \Delta(g)\check{R} \tag{27}$$

for all $g \in U_q osp(2,2)$. This means that $\check{R}$ commutes with the action of $U_q osp(2,2)$, and therefore one has a decomposition in terms of projectors.

The tensor product of two representations $[b, \frac{1}{2}]$ has the form:

$$[b, \tfrac{1}{2}] \otimes [b, \tfrac{1}{2}] = [2b, 1] \oplus [2b + \tfrac{1}{2}, \tfrac{1}{2}] \oplus [2b - \tfrac{1}{2}, \tfrac{1}{2}] \tag{28}$$

with dimensionalities 8, 4 and 4 for the right-hand-side. When $b = 0$, the two four-dimensional representations coalesce and form one eight-dimensional atypical representation (see [7, 8]). For this reason, the case $b = 0$ should be, and is, treated with some care.

I have found the eigenvalues and eigenvectors of $\check{R}(q, b)$, and then obtained the projectors form these eigenvectors. The $\check{R}$-matrix can be written as

$$\check{R}(q, b) = q^{1-4b^2} P_1(q, b) - q^{-(1+2b)^2} P_2(q, b) - q^{-(1-2b)^2} P_3(q, b) \tag{29}$$

where the P_i's form a complete set of orthogonal projectors, i.e. $P_i P_j = \delta_{ij} P_i$. They correspond to $[2b, 1]$ (P_1), $[2b + \frac{1}{2}, \frac{1}{2}]$ (P_2) and $[2b - \frac{1}{2}, \frac{1}{2}]$ (P_3). For $b = 0$, one takes a limit where the projectors P_2 and P_3 are combined. Define†

$$[x]_\pm \equiv q^x \pm q^{-x}. \tag{30}$$

† The symbol $[s]_\pm$ was also used in section 2.2 for atypical representations. However, there is no room for confusion.

I give below the non-vanishing elements of the projectors, which have a block-diagonal structure. One has:

(i) one-dimensional sub-matrices

$$P_1 = 1 \qquad P_2 = P_3 = 0 \qquad \text{for } b_i \otimes b_i,\ i = 1,2 \tag{31}$$

$$P_2 = 1 \qquad P_1 = P_3 = 0 \qquad \text{for } f_2 \otimes f_2 \tag{32}$$

$$P_3 = 1 \qquad P_1 = P_2 = 0 \qquad \text{for } f_1 \otimes f_1 \tag{33}$$

(ii) two-dimensional sub-matrices

$$\begin{aligned} P_1 &= \frac{1}{[1-2b]_+} \begin{pmatrix} q^{1-2b} & 1 \\ 1 & q^{-1+2b} \end{pmatrix} \\ P_3 &= \frac{1}{[1-2b]_+} \begin{pmatrix} q^{-1+2b} & -1 \\ -1 & q^{1-2b} \end{pmatrix} \qquad P_2 = 0 \end{aligned} \tag{34}$$

for the two bases $(b_1 \otimes f_1, f_1 \otimes b_1)$ and $(f_1 \otimes b_2, b_2 \otimes f_1)$,

$$\begin{aligned} P_1 &= \frac{1}{[1+2b]_+} \begin{pmatrix} q^{1+2b} & 1 \\ 1 & q^{-1-2b} \end{pmatrix} \\ P_2 &= \frac{1}{[1+2b]_+} \begin{pmatrix} q^{-1-2b} & -1 \\ -1 & q^{1+2b} \end{pmatrix} \qquad P_3 = 0 \end{aligned} \tag{35}$$

for the two bases $(b_1 \otimes f_2, f_2 \otimes b_1)$ and $(f_2 \otimes b_2, b_2 \otimes f_2)$,

(iii) four-dimensional sub-matrices

$$P_2(q,b) = \frac{1}{D} \begin{pmatrix} q^{-4b} & q^{-2b}\frac{[1-2b]_-^{1/2}}{[1+2b]_-^{1/2}} & q^{-2b}\frac{[1-2b]_-^{1/2}}{[1+2b]_-^{1/2}} & -1 \\ -q^{-2b}\frac{[1-2b]_-^{1/2}}{[1+2b]_-^{1/2}} & -\frac{[1-2b]}{[1+2b]} & -\frac{[1-2b]}{[1+2b]} & q^{2b}\frac{[1-2b]_-^{1/2}}{[1+2b]_-^{1/2}} \\ -q^{-2b}\frac{[1-2b]_-^{1/2}}{[1+2b]_-^{1/2}} & -\frac{[1-2b]}{[1+2b]} & -\frac{[1-2b]}{[1+2b]} & q^{2b}\frac{[1-2b]_-^{1/2}}{[1+2b]_-^{1/2}} \\ -1 & -q^{2b}\frac{[1-2b]_-^{1/2}}{[1+2b]_-^{1/2}} & -q^{2b}\frac{[1-2b]_-^{1/2}}{[1+2b]_-^{1/2}} & q^{4b} \end{pmatrix}$$

$$P_3(q,b) = P_2(q,-b) \qquad P_1(q,b) = Id_4 - P_2(q,b) - P_3(q,b)$$

where $D = [4b]_+ - 2[1-2b]/[1+2b]$, for the basis $(b_1 \otimes b_2, f_1 \otimes f_2, f_2 \otimes f_1, b_2 \otimes b_1)$. The four-dimensional pieces of the projectors depend, *a priori*, on α, β, γ and ϵ. I have chosen $\alpha = \gamma$ and $\beta = \epsilon$, which I call the symmetric choice because $\check{R}$ is a symmetric matrix, except for some minus signs in the four-dimensional pieces. Other choices for α, β, γ and ϵ, yield R-matrices that are gauge-equivalent to the foregoing matrix.

All the foregoing analysis was done using a graded tensor product. The scalar product on the space $[b, \frac{1}{2}] \otimes [b, \frac{1}{2}]$ is the graded induced scalar product, which is therefore not positive definite.

4.3. *The matrix* $\check{R}(x, q, b)$

To construct a two-dimensional lattice model one needs the spectral parameter dependent R-matrix, or the 'baxterized' form of the foregoing matrix. I find

$$\begin{aligned} \check{R}_\lambda(x,q,b) = {} & \frac{\lambda x q^{4b^2-1} + 1}{x + \lambda q^{4b^2-1}} P_1(q,b) + \frac{-\lambda x q^{(1+2b)^2} + 1}{x - \lambda q^{(1+2b)^2}} P_2(q,b) \\ & + \frac{-\lambda x q^{(1-2b)^2} + 1}{x - \lambda q^{(1-2b)^2}} P_3(q,b). \end{aligned} \tag{36}$$

The spectral parameter x is a multiplicative one (see (37)). The discrete 'index' λ in (36) can take the value q^{1-4b^2} for *all* b, or the value $-q^{-1}$ for $b=0$. Thus (36) consists of two different matrices. I think of (36) as defining one matrix, and comment on the appearance of λ in the following section. The spectral parameter dependent R-matrix is obtained as $R(x,q,b)=\mathcal{P}\check{R}(x,q,b)$. It satisfies the graded Yang–Baxter equation†

$$R_{12}(xy^{-1},q,b)R_{13}(x,q,b)R_{23}(y,q,b)=R_{23}(y,q,b)R_{13}(x,q,b)R_{12}(xy^{-1},q,b). \tag{37}$$

The R-matrices considered here are even operators. However, they are given by linear combinations of tensor products of odd and even operators, and therefore R_{13} does not act trivially on the second space (as the subscripts would imply for the non-graded case). Hence the Yang–Baxter equation is, in components,

$$R_{i_1i_2,j_1j_2}R_{j_1i_3,l_1k_3}R_{j_2k_3,l_2l_3}(-1)^{\varepsilon_{j_2}(\varepsilon_{i_3}+\varepsilon_{k_3})}=R_{i_2i_3,j_2j_3}R_{i_1j_3,k_1l_3}R_{k_1j_2,l_1l_2}(-1)^{\varepsilon_{j_2}(\varepsilon_{j_3}+\varepsilon_{l_3})} \tag{38}$$

where $\varepsilon_i=0,1$. The arguments of the matrices are the same as in equation (37). The matrix $\tilde{R}\equiv(-1)^{\varepsilon_{i_1}\varepsilon_{i_2}}R_{i_1i_2,j_1j_2}$ satisfies the ordinary Yang–Baxter equation. Therefore modulo some redefinitions the grading signs can be removed; however this does not bring any real simplifications. For $\lambda=q^{1-4b^2}$ the matrix $R(x,q,b)$ also satisfies equations (51)–(53), where the representations on the left and right of the tensor product sign $\otimes$ are both equal to $[b,\frac{1}{2}]$.

4.4. Comments

The matrix (36) has some unusual and interesting properties. This matrix is a *trigonometric* R-matrix which depends on *three* continuous and arbitrary complex parameters, x, q and b. This seems to be a new result [13]. Usually a three-parameter dependence is associated with *elliptic* solutions of the Yang–Baxter equation. In this respect the free fermion model, which depends on three complex variables, enters the context of elliptic R-matrices [14]. Trigonometric R-matrices with three parameters, for $U_qsu(2)$, were found in [15]. However, these matrices only exist at roots of unity ($q^N=1$), and correspond to *nilpotent* irreducible representations.

The dependence on the parameter b, a representation label, is immediately traced back to the non-compact generator B of the $osp(2,2)$ superalgebra. One can also obtain R-matrices which depend on an additional parameter, b', by considering an R-matrix for the representation $[b,\frac{1}{2}]\otimes[b',\frac{1}{2}]$, or for higher dimensional representations.

One can, presumably, construct trigonometric solutions that depend on a higher number of continuous parameters, if the algebra has a higher number of non-compact generators, and/or one considers representations at roots of unity.

The two matrices obtained for the two values $\lambda=q$ (for $b=0$) and $\lambda=-q^{-1}$ (for $b=0$), are *not* related by a gauge transformation. A gauge transformation is a diagonal similarity transformation; it preserves the Yang–Baxter equation. This 'doubling' occurs here at the *affine* level. The existence of two inequivalent R-matrices for the same representation was noticed for another rank two algebra, $U_qsu(3)$ [16], not $U_q\widehat{su(3)}$. However here, the two matrices may correspond to two inequivalent coproducts for the affine superalgebra $U_q\widehat{osp(2,2)}$.

The R-matrix for the fundamental representation $[b=0,\frac{1}{2}]$ of $OSp(2,2)$ was given in [5]. One can recover it by taking $\lambda=-q^{-1}$ and the limit $b=0$†. The tensor product of the two representations $[0,\frac{1}{2}]$ gives two eight-dimensional representations (as noted earlier).

† These calculations and others were carried out using MATHEMATICA™.

† This matrix does not, however, satisfy (53).

One of these representations is atypical. The limit does exist, as can be seen by carefully considering the projectors P_2 and P_3: they combine as expected to form an eight-dimensional projector onto the atypical representation, and a singlet 'projector' for the one-dimensional invariant subspace in it. The limit also exists for the other value of λ.

Similarly, the limits $b = \pm\frac{1}{2}$ are not straightforward. One can make sense of these limits from the final form (36). However, because (36) corresponds to the the 'symmetric choice', the limit corresponds to the tensor product of two direct sums $[0] \oplus [\frac{1}{2}]_{\pm}$.

I give for completeness, in the appendix, the rational limits obtained from (36). Rational limits can be obtained by letting both x and q tend to one, with x behaving as some power of q.

I have obtained new trigonometric R-matrices which depend on three parameters. I believe these matrices have features which were not found before in the literature.

5. Bethe ansatz

5.1. The need for fusion

The matrix (36) has exactly 36 non-vanishing matrix elements. The corresponding 36-vertex model cannot be solved using a direct Bethe ansatz approach. There are simply too many non-vanishing matrix elements. This is due to the fact that the representation $[b, \frac{1}{2}]$ is not the smallest representation of the algebra $osp(2,2)$. Because of the 'excess' of matrix elements, the algebraic RTT relations, which play a central role in the algebraic Bethe ansatz method, do not have the form which allow a Bethe ansatz. For this same reason too, the obvious candidates for 'highest weights vectors', from which the Bethe ansatz eigenvector is built by applying lowering operators, do not have the required properties. Namely, there are too many elements of the monodromy matrix, (41), which do not annihilate these candidates. The situation is similar for higher dimensional representations of q-deformed Lie algebra. A fusion procedure is used for all those cases. See [12] for $U_q su(2)$ for instance.

In what follows I take $\lambda = q^{1-4b^2}$ (in (36)).

5.2. Auxiliary and quantum spaces

Fusion requires involving other lattice models. To do this, one finds solutions of the Yang–Baxter equation (the dependence on the other parameters is implicit)

$$R_{12}(xy^{-1})R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(xy^{-1}) \tag{39}$$

where the three spaces involved are not necessarily copies of the same space. The matrices R exist for the tensor product of any representations of a given q-deformed Lie algebra, or superalgebra. The computations needed to find these matrices are not usually simple. Now recall how one obtains transfer matrices of vertex models. Define then the local operator $L_n(x)$, at site n and for an auxiliary space denoted by 'a', by

$$L_n(x) = R_{\mathrm{a}n}(x). \tag{40}$$

The corresponding monodromy matrix, $T(x)$, is

$$T(x) = L_L(x) \otimes L_{L-1}(x) \otimes \ldots \otimes L_1(x) \tag{41}$$

where the tensor product is graded, and L is the number of sites. The transfer matrix is given by

$$\tau(x) = \mathrm{Str_a}(T(x)) \equiv \sum_a (-1)^{\varepsilon_a} T_{aa}(x) \tag{42}$$

where Str_a is the supertrace over the auxiliary space 'a'. I am considering here periodic boundary conditions. I shall come back to this point later.

Equations (39), (40) and (41) imply that the monodromy matrices intertwine according to

$$R_{12}(xy^{-1}) \overset{1}{T}(x) \overset{2}{T}(y) = \overset{2}{T}(y) \overset{1}{T}(x) R_{12}(xy^{-1}) \tag{43}$$

where $\overset{1}{T}(x) = T(x) \otimes 1$ and $\overset{2}{T}(y) = 1 \otimes T(y)$.

The two auxiliary spaces, 1 and 2 in the subscripts and 'top-scripts' of (43), are not necessarily the same. Both transfer matrices act in the same quantum space, which is space 3 in (39) tensored L times. The supertrace over the two auxiliary spaces 1 and 2, of the RTT relations (43), gives the commutations of the transfer matrices at different spectral parameters:

$$[\tau_1(x), \tau_2(y)] = 0. \tag{44}$$

The subscripts in (44) are just to remind us that the two auxiliary spaces can be different. This commutation relation implies that the transfer matrices can be diagonalized *simultaneously*.

5.3. Fusion

The second ingredient in the fusion procedure consists of obtaining relations between some transfer matrices that commute, as in (44). An R-matrix 'degenerates', or becomes block-diagonally proportional to projectors, at certain values of the spectral parameter x. The projections are on representations which appear in the tensor product of the representations, for which the R-matrix was built. This degeneracy is a generic feature of R-matrices.

The projectors are used to construct R-matrices for the representations corresponding to the projectors. I give explicit examples of such a degeneracy and fusion in the following sections.

5.4. Some technical considerations

I follow the philosophy of [12]. One wants to diagonalize, by algebraic Bethe ansatz, the transfer matrix of the model constructed from the R-matrix of $V_1 \otimes V_2$, for two representations V_1 and V_2 of a certain Lie algebra (or superalgebra). If V_1, the auxiliary space, is the 'fundamental' representation of the algebra then a diagonalization by algebraic Bethe ansatz can be performed without further ado. This is because the monodromy matrices intertwine with the R-matrix of the tensor product of two fundamental representations. If V_1 is not 'the' fundamental, it should obtained by tensor product of the fundamental and smaller representations. The Bethe ansatz is performed for (fund. rep.) $\otimes V_2$. The eigenvalues, for the transfer matrix of the lattice model $V_1 \otimes V_2$, are obtained using the fusion equations and the commutation of the intermediate transfer matrices.

Note that higher representations can be considered. They give relations between transfer matrices, but are not really helpful for doing a Bethe ansatz.

The $osp(2, 2)$ algebra has two three-dimensional representations, $[\frac{1}{2}]_+$ and $[\frac{1}{2}]_-$. These are the smallest non-trivial representations. I consider $[\frac{1}{2}]_+$. As we shall see, the R-matrix of $[\frac{1}{2}]_+ \otimes [\frac{1}{2}]_+$ has a form which allows an algebraic Bethe ansatz. To 'bridge the gap' between $[\frac{1}{2}]_+$ and $[b, \frac{1}{2}]$, I determine the R-matrix for $[\frac{1}{2}]_+ \otimes [b, \frac{1}{2}]$. The monodromy matrices of this tensor product intertwine using the transfer matrix of $[\frac{1}{2}]_+ \otimes [\frac{1}{2}]_+$. An algebraic Bethe ansatz for the transfer matrix of $[\frac{1}{2}]_+ \otimes [b, \frac{1}{2}]$ is possible. I do this in section 5.8. Then, to remain faithful to the foregoing philosophy, one would like to obtain the representation $[b, \frac{1}{2}]$ in a tensor product of representations $[\frac{1}{2}]_\pm$. However, after careful consideration, the only relevant tensor product is of $[\frac{1}{2}]_+$ with itself:

$$[\tfrac{1}{2}]_+ \otimes [\tfrac{1}{2}]_+ = [\tfrac{3}{2}, \tfrac{1}{2}] \oplus [1]_+ \tag{45}$$

with dimensions 4 and 5 for the right-hand side.

The representation theory of $osp(2,2)$ is such that, of all the four-dimensional representations $[b, \frac{1}{2}]$, only $[\pm\frac{3}{2}, \frac{1}{2}]$ can be obtained in the tensor product of the three-dimensional representations. In this sense, the representations $[b, \frac{1}{2}]$ are 'fundamental' too.

Nevertheless, a fusion in the auxiliary space to obtain the transfer matrix of $[\frac{3}{2}, \frac{1}{2}] \otimes [b, \frac{1}{2}]$, the eigenvalues, and the Bethe ansatz equations, is useful. I use these results to conjecture the eigenvalues and Bethe ansatz equations of the R-matrix of $[b, \frac{1}{2}] \otimes [b, \frac{1}{2}]$.

Let $3 = [\frac{1}{2}]_+$, $4' = [\frac{3}{2}, \frac{1}{2}]$ and $4 = [b, \frac{1}{2}]$ to simplify the notation. One then has five R-matrices to consider, $R^{3\otimes 3}$, $R^{3\otimes 4'}$, $R^{3\otimes 4}$, $R^{4'\otimes 4}$ and $R^{4\otimes 4}$.

In what follows the spectral parameters are arbitrary. For the first and second spaces equal to the three-dimensional representation, and the third space equal to $[b, \frac{1}{2}]$, in (39), one obtains a monodromy matrix with a $[\frac{1}{2}]_+$-auxiliary space and a $[b, \frac{1}{2}]$-quantum space. These $T^{(34)}$ monodromy matrices intertwine as in (43) with the matrix $R^{3\otimes 3}$, and therefore their transfer matrices, which act in the tensor products of $[b, \frac{1}{2}]$-spaces, commute. It is these transfer matrices that are directly diagonalized by Bethe ansatz. A first space equal to $[\frac{1}{2}]_+$, a second space equal to $[\frac{3}{2}, \frac{1}{2}]$, and a third space equal to $[b, \frac{1}{2}]$ give monodromy matrices $T^{(34)}$ and $T^{(4'4)}$ intertwining according to a $[\frac{1}{2}]_+ \otimes [\frac{3}{2}, \frac{1}{2}]$-matrix. The transfer matrices again commute. A fusion equation (58) is then found. It relates the relevant eigenvalues. Similarly, considering $(1,2,3) = (4', 4, 4)$, in an obvious notation, ensures the possibility of simultaneous diagonalization of the $\tau^{(4'4)}$ and $\tau^{(44)}$ matrices. Considering $(1,2,3) = (4', 4', 4)$ ensures the possibility of simultaneous diagonalization of $\tau^{(4'4)}$ matrices. Finally, considering $(1,2,3) = (4,4,4)$ ensures the possibility of simultaneous diagonalization of $\tau^{(44)}$ matrices.

5.5. *The R-matrix for* $[\frac{1}{2}]_+ \otimes [\frac{1}{2}]_+$

Let $B_1 = |\frac{1}{2}, \frac{1}{2}\rangle$, $F = |1, 0\rangle$ and $B_2 = |\frac{1}{2}, -\frac{1}{2}\rangle$ form the basis of $[\frac{1}{2}]_+$. These vectors should not be confused with the four vectors defined earlier. The q-deformed generators are

$$P_+ = \mathrm{diag}(0, -1, -1) \qquad P_- = \mathrm{diag}(1, 1, 0) \tag{46}$$

$$(V_+)_{23} = \alpha' \qquad (V_-)_{21} = -\alpha' \qquad (\overline{V}_+)_{12} = \gamma' \qquad (\overline{V}_-)_{32} = -\gamma' \tag{47}$$

where only the non-vanishing elements of the odd generators are given. The multiplets normalizations α' and γ' satisfy

$$4\alpha'\gamma' = [2]. \tag{48}$$

The corresponding R-matrix, denoted by r, is found using methods similar to those used to determine the R-matrix of the four-dimensional representation. The matrix r depends only on the product $\alpha'\gamma'$, and no choice of α' and γ' is necessary. The non-vanishing elements of r are given by

$$r = 1 \qquad \text{for } B_1 \otimes B_1,\ B_2 \otimes B_2$$

$$r = \frac{xq^4 - 1}{x - q^4} \qquad \text{for } F \otimes F$$

$$r = \begin{pmatrix} j & i \\ h & j \end{pmatrix}$$

for the bases $(B_1 \otimes F, F \otimes B_1)$, $(B_1 \otimes B_2, B_2 \otimes B_1)$, $(F \otimes B_2, B_2 \otimes F)$, where

$$h(x,q) = \frac{1-q^4}{x-q^4} \qquad i(x,q) = xh(x,q) \qquad j(x,q) = q^2 \frac{x-1}{x-q^4}. \tag{49}$$

5.6. The R-matrix for $[\frac{1}{2}]_+ \otimes [b, \frac{1}{2}]$

The determination of the 'hybrid' R-matrix, corresponding to a left representation $[\frac{1}{2}]_+$ and a right representation $[b, \frac{1}{2}]$, proceeds differently. Because the representations are different, the commutation of the coproduct Δ with an $\check{R}$-matrix does not hold. Instead one can still use the universal spectral parameter independent R-matrix with the two foregoing representations to obtain the form of the matrix and hence simplify the following calculation. Jimbo [17] has shown for affine q-deformed Lie algebras that a solution of

$$\overline{\hat{\Delta}}(g)R(x) = R(x)\hat{\Delta}(g) \tag{50}$$

for the generators g of the *affine* q-deformed algebra $U_q\widehat{osp(2,2)}$, automatically satisfies the Yang–Baxter equation. This result was generalized in [18] for superalgebras. The coproducts $\overline{\hat{\Delta}}$ and $\hat{\Delta}$ extend the coproducts defined earlier to the affine algebra $U_q\widehat{osp(2,2)}$. The affine coproducts coincide with the usual ones for all the generators of $U_qosp(2,2)$ except for the additional three generators belonging to the affine algebra. The matrix $R^{3\otimes 4}(x,q,b)$ has the same block-diagonal structure as the corresponding spectral parameter independent matrix. More precisely the matrix acts non-trivially in the same subspaces. I denote $R^{3\otimes 4}$ by R in this section and in the following sections. It is enough to look for a solution of the following equations:

$$R(x)\Delta(V_+) = \overline{\Delta}(V_+)R(x) \tag{51}$$

$$R(x)\Delta(\overline{V}_+) = \overline{\Delta}(\overline{V}_+)R(x) \tag{52}$$

$$R(x)[e_0 \otimes q^{h_0/2} + xq^{-h_0/2} \otimes e_0] = [e_0 \otimes q^{-h_0/2} + xq^{h_0/2} \otimes e_0]R(x) \tag{53}$$

where $h_0 = 4S_3$ and e_0 is proportional to the q-deformed generator S_- (the normalization is irrelevant). With f_0 proportional to the q-deformed generator S_+, the three generators e_0, f_0 and h_0 correspond to the additional root of the q-deformed affine superalgebra $U_q\widehat{osp(2,2)}$. An equation similar to (53) holds for f_0. Note that, for the representations at hand, the generators on the left of the tensor products are in the three-dimensional representation while those on the right are in the four-dimensional one. This system of equations is overdetermined in the 22 unknowns of $R(x)$. This is always the case for such systems. I take $\alpha = \gamma$, $\beta = \epsilon$ in (21), (22), and $\alpha' = \gamma'$ in (47). Solving the linear equations (51)–(53), I obtain the non-vanishing elements of the 12×12 R-matrix:

$$R = q^{1-2b} \qquad \text{for } B_1 \otimes b_1,\ B_1 \otimes f_1,\ B_2 \otimes f_1,\ B_2 \otimes b_2$$

$$R = \frac{xq^4 - q^{1-2b}}{x - q^{3+2b}} \qquad \text{for } F \otimes f_2$$

$$R = \begin{pmatrix} \frac{xq^{3-2b}-q^2}{x-q^{3+2b}} & -\frac{q^{\frac{5}{2}-b}[1]_+^{1/2}[1]_-^{1/2}[1+2b]_-^{1/2}x}{x-q^{3+2b}} \\ -\frac{q^{\frac{5}{2}-b}[1]_+^{1/2}[1]_-^{1/2}[1+2b]_-^{1/2}}{x-q^{3+2b}} & \frac{xq^2-q^{3-2b}}{x-q^{3+2b}} \end{pmatrix}$$

for the basis $(B_1 \otimes f_2, F \otimes b_1)$,

$$R = \begin{pmatrix} \frac{xq^2 - q^{3-2b}}{x - q^{3+2b}} & -\frac{q^{\frac{5}{2}-b}[1]_+^{1/2}[1]_-^{1/2}[1+2b]_-^{1/2}x}{x-q^{3+2b}} \\ -\frac{q^{\frac{5}{2}-b}[1]_+^{1/2}[1]_-^{1/2}[1+2b]_-^{1/2}}{x-q^{3+2b}} & \frac{xq^{3-2b}-q^2}{x-q^{3+2b}} \end{pmatrix}$$

for the basis $(F \otimes b_2, B_2 \otimes f_2)$,

$$R = \begin{pmatrix} \frac{xq^{3-2b}-q^2}{x-q^{3+2b}} & \frac{q^{\frac{3}{2}-b}[1-2b]_-^{1/2}[1]_+^{1/2}[1]_-^{1/2}x}{x-q^{3+2b}} & -\frac{xq^2[2]_-}{x-q^{3+2b}} \\ -\frac{q^{\frac{7}{2}-b}[1-2b]_-^{1/2}[1]_+^{1/2}[1]_-^{1/2}}{x-q^{3+2b}} & \frac{x-q^{5-2b}}{x-q^{3+2b}} & \frac{q^{\frac{3}{2}-b}[1-2b]_-^{1/2}[1]_+^{1/2}[1]_-^{1/2}x}{x-q^{3+2b}} \\ -\frac{q^{3-2b}[2]_-}{x-q^{3+2b}} & -\frac{q^{\frac{7}{2}-b}[1-2b]_-^{1/2}[1]_+^{1/2}[1]_-^{1/2}}{x-q^{3+2b}} & \frac{xq^{3-2b}-q^2}{x-q^{3+2b}} \end{pmatrix}$$

for the basis $(B_1 \otimes b_2, F \otimes f_1, B_2 \otimes b_1)$. The matrix $R^{3\otimes 4}$ has the following properties:

$$R(0,q,b) = R(q,b) \qquad R^{-1}(x,q,b) = q^{4b-2}R^{\mathrm{ST}}(x^{-1},q,b) \tag{54}$$

where $R(q,b)$ is the spectral parameter independent R-matrix and the superscript 'ST' denotes a supertransposition for the non-positive definite scalar product.

5.7. *A fusion equation*

The next step in obtaining Bethe ansatz equations consists of obtaining a fusion equation that will give a functional relation between the eigenvalues of the four-dimensional model and those of the hybrid model. First one notices that the three-dimensional R-matrix becomes block-diagonally proportional to a five-dimensional projector for $x = q^{-4}$. Define a projector P as

$$P = Ar(q^{-4}, q) \qquad \text{with } A = \mathrm{diag}\left(1, \frac{a}{2}, \frac{a}{2}, \frac{a}{2}, 1, \frac{a}{2}, \frac{a}{2}, \frac{a}{2}, 1\right) \tag{55}$$

where $a = q^2 + q^{-2}$. Let $p \equiv 1 - P$. One can now do the fusion in the auxiliary space. Define the matrix

$$\tilde{R}_{(12)3}(x,q,b) = B_{12}p_{12}R_{13}(xq^{-2},q,b)R_{23}(xq^2,q,b)p_{12}B_{12}^{-1} \tag{56}$$

where the nine-dimensional matrix B is the change of basis matrix, to the basis that diagonalizes both projectors, P and p. B is given by three ones and three blocks

$$\begin{pmatrix} -q^2 & 1 \\ q^2 & 1 \end{pmatrix}. \tag{57}$$

The matrix $\tilde{R}$ has 36^2 elements. However, there are many rows and columns of zeros. One can remove them and obtain a 16×16 R-matrix for $[\frac{3}{2}, \frac{1}{2}] \otimes [b, \frac{1}{2}]$. The matrix (56), which was denoted $R^{4' \otimes 4}$ in section 5.4, satisfies the various Yang–Baxter equations discussed in the same section.

The supertrace, in the spaces 1 and 2, of equation (56) replicated to an L-site chain, gives a relation between the transfer matrices of the $[\frac{1}{2}]_+ \otimes [b, \frac{1}{2}]$, $[\frac{3}{2}, \frac{1}{2}] \otimes [b, \frac{1}{2}]$ and $[1]_+ \otimes [b, \frac{1}{2}]$ lattices:

$$-\tau^{(4'4)}(x) = \tau^{(34)}(xq^{-2})\tau^{(34)}(xq^2) + \tau^{(54)}(x). \tag{58}$$

The superscripts refer to the dimensions of the representations. The additional minus sign accounts for a mismatch in the grading of the bases. As explained in section 5.4, all the transfer matrices in (58) commute for all values of the spectral parameter x. Therefore the same relation holds for the respective eigenvalues.

5.8. Bethe ansatz

I now perform a nested algebraic Bethe ansatz calculation to diagonalize the transfer matrix of the the $3 \otimes 4$ model (see [19] for instance). The matrix L_n defined above is given by (40) and $R^{3\otimes 4}(x)$. In components L_n reads

$$L(u)_{a\alpha,b\beta} = R^{3\otimes 4}(u)_{a\alpha,b\beta} \qquad 1 \leqslant a, b \leqslant 3 \qquad 1 \leqslant \alpha, \beta \leqslant 4 \tag{59}$$

where the Latin indices correspond to the auxiliary space, and the Greek indices to the quantum space. The monodromy matrix (41) satisfies (as explained in section 5.4)

$$\check{r}_{12}(u-v)T(u) \otimes T(v) = T(v) \otimes T(u)\check{r}_{12}(u-v). \tag{60}$$

In components, with all Latin indices varying from 1 to 3, and repeated indices indicating summations, (60) reads:

$$\begin{aligned}\check{r}_{d_1d_2,b_1b_2}(u-v)T_{b_1c_1}(u)T_{b_2c_2}(v)(-1)^{\varepsilon_{b_2}(\varepsilon_{b_1}+\varepsilon_{c_1})} \\ = T_{d_1b_1}(v)T_{d_2b_2}(u)\check{r}_{b_1b_2,c_1c_2}(u-v)(-1)^{\varepsilon_{d_2}(\varepsilon_{d_1}+\varepsilon_{b_1})} \qquad \varepsilon_1 = \varepsilon_3 = 0,\ \varepsilon_2 = 1.\end{aligned} \tag{61}$$

The monodromy matrix (41), in components, is given by

$$\begin{aligned}(T(u)^{ab})_{\alpha_1,\ldots,\alpha_L;\beta_1,\ldots,\beta_L} &= L(u)_{a\alpha_L,c_L\beta_L}L(u)_{c_L\alpha_{L-1},c_{L-1}\beta_{L-1}}\cdots \\ &\quad \ldots L(u)_{c_2\alpha_1,b\beta_1}(-1)^{\sum_{j=2}^{L}(\varepsilon_{\alpha_j}+\varepsilon_{\beta_j})\sum_{i=1}^{j-1}\varepsilon_{\alpha_i}}.\end{aligned} \tag{62}$$

The signs arise because of the graded tensor product.

There are two obvious candidates for the 'highest weight vector', or ferromagnetic vacuum, $|\Omega_1\rangle = f_1^{(1)} \otimes \ldots \otimes f_1^{(L)}$ and $|\Omega_2\rangle = f_2^{(1)} \otimes \ldots \otimes f_2^{(L)}$, where f_1 and f_2 were defined in (20). The two vectors constructed out of b_1, or b_2, turn out not to be annihilated by enough T_{ij}'s to perform the ansatz. In what follows I give the results for $|\Omega_1\rangle$. I shall use the following notation for the matrix of operators T:

$$T = \begin{pmatrix} D_{11} & C_1 & D_{13} \\ B_1 & A & B_3 \\ D_{31} & C_3 & D_{33} \end{pmatrix}. \tag{63}$$

The dependence on u, or $x = e^{iu}$, is implicit. The action of T on $|\Omega_1\rangle$ can be summarized in

$$T|\Omega_1\rangle = \begin{pmatrix} (R_{22})^L & C_1 & 0 \\ 0 & (R_{66})^L & 0 \\ 0 & C_3 & (R_{10\,10})^L \end{pmatrix} |\Omega_1\rangle. \tag{64}$$

Recall that the R-matrix elements are those of $R^{3\otimes 4}(x)$. The fermionic 'creation' operators are given by $C_1 = T_{12}$ and $C_3 = T_{32}$. These operators change the spin of the vector they are acting on by units of $-\frac{1}{2}$ and $\frac{1}{2}$, respectively. The eigenvector ansatz is

$$|\underline{u}, F\rangle = C_{a_1}(u_1)C_{a_2}(u_2)\ldots C_{a_n}(u_n)|\Omega_1\rangle F^{a_n \ldots a_2 a_1} \qquad a_i \in \{1, 3\} \tag{65}$$

where the spectral parameters $\underline{u}$ and the coefficients F are to be determined. The transfer matrix is

$$\tau = \mathrm{Str}_a(T) = T_{11} + T_{33} - T_{22}. \tag{66}$$

Note that a twist on the periodic boundary conditions can be introduced at this level.

The calculation now proceeds along the usual lines of the nested algebraic Bethe ansatz method. One pushes the three pieces of the transfer matrix through the creation operators of the eigenvector, and obtains 'wanted' and 'unwanted' contributions. The former contributions are proportional to the eigenvector ansatz, while the latter are forced to vanish, giving a condition on the coefficients F. More precisely F has to be an eigenvector of a spin-chain on n sites, with inhomogeneities given by the u_i's, and constructed out of a 4×4 R-matrix obtained from the nine-dimensional matrix r. By doing a similar ansatz, one obtains the eigenvectors of this chain, with one set of equations. Requiring F to be an eigenvector gives another set of equations.

5.9. Eigenvalues and Bethe ansatz equations

The eigenvalues of the 'hybrid' model can be written:

$$\Lambda^{3\otimes 4}(x) = (R_{22})^L \prod_{j=1}^{m} \frac{1}{j(y_j x^{-1})} + (R_{22})^L \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{1}{j(x_i x^{-1}) j(x y_j^{-1})} - (R_{66}(x))^L \prod_{i=1}^{n} \left(\frac{-g(x x_i^{-1})}{j(x x_i^{-1})} \right) \tag{67}$$

where the parameters x_i and y_j are solutions of the Bethe ansatz equations

$$\left(\frac{R_{66}(x_k)}{R_{22}}\right)^L = \prod_{j=1}^{m} \frac{1}{j(x_k y_j^{-1})} \qquad i = 1, \ldots, n \tag{68}$$

$$\prod_{i=1}^{n} j(x_i y_l^{-1}) = \prod_{\substack{j=1 \\ j\neq l}}^{m} \frac{j(y_j y_l^{-1})}{j(y_l y_j^{-1})} \qquad l = 1, \ldots, m. \tag{69}$$

These Bethe ansatz equations reflect the choice of a particular grading of the bases.

The R-matrix of the lattice model with auxiliary space $[\frac{3}{2}, \frac{1}{2}]$ and quantum space $[b, \frac{1}{2}]$ and the eigenvalues of the corresponding transfer matrix are obtained from the fusion equation (58). I find for the eigenvalues (up to an unessential factor $q^{(2-4b)L}$):

$$\begin{aligned}\Lambda^{4'\otimes 4}(x) = \prod_{i=1}^{n} \frac{x_i q^{-3} - x q^3}{x_i q^{-1} - x q} - \left(\frac{x q^{-3/2+b} - q^{3/2-b}}{x q^{-1/2-b} - q^{1/2+b}}\right)^L \\ \times \left(\prod_{i=1}^{n}\prod_{j=1}^{m} \frac{x q - y_j q^{-1}}{x q^{-1} - y_j q} \frac{x q^3 - x_i q^{-3}}{x q - x_i q^{-1}} + \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{x q^{-3} - y_j q^3}{x q^{-1} - y_j q} \frac{x q^3 - x_i q^{-3}}{x q^{-1} - x_i q} \right) \\ + \left(\frac{x q^{b-3/2} - q^{3/2-b}}{x q^{-1/2-b} - q^{b+1/2}} \frac{x q^{b-7/2} - q^{7/2-b}}{x q^{-5/2-b} - q^{5/2+b}} \right)^L \prod_{i=1}^{n} \frac{x q^3 - x_i q^{-3}}{x q^{-1} - x_i q}.\end{aligned} \tag{70}$$

Recall that $4' \equiv [\frac{3}{2}, \frac{1}{2}]$ (the auxiliary space) and $4 \equiv [b, \frac{1}{2}]$ (the quantum space). Both eigenvalues (67) and (70) seem to have poles at values related to the ansatz parameters, x_i and y_j. However, the Bethe ansatz equations ensure precisely that the residues, at these apparent poles, vanish. This is generally the case with a Bethe ansatz solution. It is also possible to obtain from (58) the eigenvalues of the transfer matrix based on the R-matrix of the representation $[1]_+ \otimes [b, \frac{1}{2}]$.

I now rewrite the Bethe ansatz equations, (68) and (69), in the 'generic form'. For $x = e^{iu}$, $q = e^{i\gamma/2}$ and some redefinitions, the equations become†:

$$\left(\frac{\sinh\frac{1}{2}(u_k - (b-1/2)i\gamma)}{\sinh\frac{1}{2}(u_k + (b-1/2)i\gamma)}\right)^L = \prod_{j=1}^{m} \frac{\sinh\frac{1}{2}(u_k - v_j + i\gamma)}{\sinh\frac{1}{2}(u_k - v_j - i\gamma)} \qquad 1 \leqslant k \leqslant n \tag{71}$$

† This γ should not be confused with the one entering the generators of $osp(2,2)$ for the representation $[b, \frac{1}{2}]$.

$$\prod_{k=1}^{n}\frac{\sinh\frac{1}{2}(u_k-v_j+\mathrm{i}\gamma)}{\sinh\frac{1}{2}(u_k-v_j-\mathrm{i}\gamma)}=-\prod_{l=1}^{m}\frac{\sinh\frac{1}{2}(v_l-v_j+2\mathrm{i}\gamma)}{\sinh\frac{1}{2}(v_l-v_j-2\mathrm{i}\gamma)}\qquad 1\leqslant j\leqslant m. \tag{72}$$

The choice of the ferromagnetic vacuum $|\Omega_2\rangle$ for the eigenvector ansatz gives a different set of equations. They differ from (71)–(72) by the change $b-\frac{1}{2}\to-(b+\frac{1}{2})$. The eigenvalues also differ in this manner. Such a difference between the Bethe ansatz results indicate that the eigenvectors obtained from both $|\Omega_1\rangle$ and $|\Omega_2\rangle$ may be necessary to have a complete set of eigenvectors. This will be investigated elsewhere.

5.10. A conjecture

As indicated earlier, the representation $[b,\frac{1}{2}]$ cannot be obtained from a tensor product of smaller representations (if $b\neq\pm\frac{3}{2}$). I conjecture the result for the eigenvalues of $\tau^{(44)}$ transfer matrix, based on the Bethe ansatz equations already obtained, and which also hold for $4\otimes 4$ since they depend only on the quantum space and the algebra, and other considerations. More precisely, I replace the prefactors raised to the power L in (70), by the ones corresponding to the matrix $\tau^{4\otimes 4}$, acting on the eigenvector $|\Omega_1\rangle$. Then this eigenvalue is 'dressed' by products obtained through a 'minimal' modification of the terms inside the products of the eigenvalue (70). This modification must preserve the Bethe ansatz equations found earlier, meaning, the vanishing of the residues at the apparent poles must yield the same equations. Finally the eigenvalues of the Hamiltonian must be generically real. The eigenvalues are then given by

$$\begin{aligned}
\Lambda^{4\otimes 4}(u)={}&\left(\frac{\sin\frac{1}{2}(u+(1-2b)\gamma)}{\sin\frac{1}{2}(u-(1-2b)\gamma)}\right)^{L}\prod_{k=1}^{n}\frac{\sinh\frac{1}{2}(u\mathrm{i}+u_k+(b-\frac{1}{2})\gamma\mathrm{i})}{\sinh\frac{1}{2}(u\mathrm{i}+u_k-(b-\frac{1}{2})\gamma\mathrm{i})}\\
&-\left(\frac{\sin(\frac{u}{2})}{\sin\frac{1}{2}(u-(1-2b)\gamma)}\right)^{L}\\
&\left(\prod_{k=1}^{n}\prod_{l=1}^{m}\frac{\sinh\frac{1}{2}(u\mathrm{i}+v_l+(\frac{3}{2}-b)\gamma\mathrm{i})\,\sinh\frac{1}{2}(u\mathrm{i}+u_k+(b-\frac{1}{2})\gamma\mathrm{i})}{\sinh\frac{1}{2}(u\mathrm{i}+v_l-(\frac{1}{2}+b)\gamma\mathrm{i})\,\sinh\frac{1}{2}(u\mathrm{i}+u_k-(b-\frac{1}{2})\gamma\mathrm{i})}\right.\\
&\left.+\prod_{k=1}^{n}\prod_{l=1}^{m}\frac{\sinh\frac{1}{2}(u\mathrm{i}+v_l-(\frac{5}{2}+b)\gamma\mathrm{i})\,\sinh\frac{1}{2}(u\mathrm{i}+u_k+(b-\frac{1}{2})\gamma\mathrm{i})}{\sinh\frac{1}{2}(u\mathrm{i}+v_l-(\frac{1}{2}+b)\gamma\mathrm{i})\,\sinh\frac{1}{2}(u\mathrm{i}+u_k-(\frac{3}{2}+b)\gamma\mathrm{i})}\right)\\
&+\left(\frac{\sin(\frac{u}{2})\sin\frac{1}{2}(u-2\gamma)}{\sin\frac{1}{2}(u-(1+2b)\gamma)\sin\frac{1}{2}(u-(1-2b)\gamma)}\right)^{L}\\
&\times\prod_{k=1}^{n}\frac{\sinh\frac{1}{2}(u\mathrm{i}+u_k+(b-\frac{1}{2})\gamma\mathrm{i})}{\sinh\frac{1}{2}(u\mathrm{i}+u_k-(\frac{3}{2}+b)\gamma\mathrm{i})}.
\end{aligned}\tag{73}$$

The integers n and m can be restricted to

$$0\leqslant m\leqslant n\leqslant 3L \tag{74}$$

the ansatz eigenvectors would otherwise identically vanish. This follows from an analysis of the action of the creation operators on the generating vectors at both levels of the Bethe ansatz.

Considering higher dimensional representations, and the relations between the respective transfer matrices, one can obtain functional relations for the eigenvalues of the transfer matrices, including the transfer matrix of the lattice with both auxiliary and quantum spaces equal to $[b,\frac{1}{2}]$. These relations can serve as a check for (73).

5.11. Hamiltonian and momentum

The logarithmic derivative of the transfer matrix $\frac{1}{i}\tau^{4\otimes 4}$ at $x = 1$ or $u = 0$ gives a spin-chain Hamiltonian. This Hamiltonian has coupling constants that depend on γ and b. However, the eigenvalues can be real for certain excitations. Dropping the contribution from the prefactor, which amounts to a translation of the energy origin, the eigenvalues of the Hamiltonian are given by

$$E = \frac{1}{2}\sum_{k=1}^{n} \frac{\sin((\frac{1}{2}-b)\gamma)}{\sinh\frac{1}{2}(u_k + (\frac{1}{2}-b)\gamma \mathrm{i})\sinh\frac{1}{2}(u_k - (\frac{1}{2}-b)\gamma \mathrm{i})}. \tag{75}$$

The eigenvalues of the momentum operator, $P = \frac{1}{i}\ln\tau(u=0)$, are given by

$$P = \frac{1}{i}\sum_{k=1}^{n}\ln\left(\frac{\sinh\frac{1}{2}(u_k + \gamma(b-\frac{1}{2})\mathrm{i})}{\sinh\frac{1}{2}(u_k - \gamma(b-\frac{1}{2})\mathrm{i})}\right) \tag{76}$$

up to a translation by a constant.

5.12. The parameter b: a generalized spin

The Bethe ansatz equations (71) and (72) have the form expected from the algebra and the highest weight label of the representation $[b, \frac{1}{2}]$. The $b - \frac{1}{2}$ appearing in the left-hand-side of (71) is proportional to the scalar product of this highest weight and the first root of the simple root basis.

The comparison of the Bethe ansatz equations, (71) and (72), with those of the $SU(2)$-chain with arbitrary spin, shows why $b - \frac{1}{2}$ can be considered as some equivalent of a continuous spin label. In contrast to the generalized spin of [15], which only exist for q a root of unity, the 'spin' $b - \frac{1}{2}$ exist for all values of q.

For b in certain ranges with rational bounds, the value of the central charge is independent of b; however the values of the central charge will be different for each domain. The conformal weights will depend on b continuously throughout each domain.

6. Conclusion

The motivation for studying lattice models with an underlying $U_q osp(2,2)$ symmetry arose from an attempt at constructing lattice models which in a certain continuum limit could yield $N = 2$ superconformal field theories. The supersymmetry may then be traced back to the lattice. The first step in such an approach consists of determining R-matrices. In this paper I have derived trigonometric R-matrices which depend on two continuous generic parameters q and b. The origin of the second parameter is a $U(1)$ generator in the Cartan subalgebra†. I give the rational limits of the matrices obtained; they also depend on the parameter b. A Bethe ansatz diagonalization of the transfer matrix is complicated by the representation theory of the superalgebra. After obtaining the Bethe ansatz equations for a specific model using fusion I conjecture the result for the eigenvalues of the $[b, \frac{1}{2}] \otimes [b, \frac{1}{2}]$ lattice. These results can readily be generalized to twisted periodic boundary conditions. Preliminary results indicate that the central charge in the continuum limit does not depend continuously on b (the conformal weights do however). These results depend on a careful analysis of the Bethe ansatz equations. The determination of the vacuum of the model is

† R-matrices with a double parametric dependence have been derived for $U_q su(2)$ [15]; however, there the deformation parameter q has to be a root of unity.

complicated by the grading of the algebra†. A numerical study will be useful to find the ground state. Transfer matrices with periodic boundary conditions as studied here commute with each other at different values of the spectral parameter, but they do not commute with the generators of $U_q osp(2,2)$. This does not mean, however, that such a symmetry is not present. For $SU(2)$ the underlying symmetry was exhibited in [21] for twisted boundary conditions. This can presumably be generalized. However, by considering open boundary conditions it is possible to obtain transfer matrices which commute with the q-deformed algebra. To obtain such matrices one needs to find K-matrices. This is part of a work in progress.

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Appendix

I briefly discuss the rational limits [22] obtained from (36). Rational limits can be obtained by letting both x and q tend to one, with x behaving as some power of q. Consider for instance the limit obtained for the choices $\lambda = q^{1-4b^2}$, $q = \mathrm{e}^{-\mathrm{i}\gamma}$ and $x = \mathrm{e}^{4\mathrm{i}u\gamma}$, where γ tends to zero. The resulting rational R-matrix is

$$\check{R}(u,b) = P_1(q=1,b) + g_2(u,b)P_2(q=1,b) + g_3(u,b)P_3(q=1,b) \quad (77)$$

where

$$g_2(u,b) = \frac{1+2b-2u}{1+2b+2u} \qquad g_3(u,b) = \frac{1-2b-2u}{1-2b+2u}. \quad (78)$$

The block-diagonal matrix (77) contains:

(i) one-dimensional sub-matrices

$$\check{R}(u,b) = 1 \qquad \text{for } b_i \otimes b_i,\ i = 1,2 \quad (79)$$

$$\check{R}(u,b) = g_2(u,b) \qquad \text{for } f_2 \otimes f_2 \quad (80)$$

$$\check{R}(u,b) = g_3(u,b) \qquad \text{for } f_1 \otimes f_1 \quad (81)$$

(ii) two-dimensional sub-matrices

$$\check{R}(u,b) = \frac{1}{1-2b+2u}\begin{pmatrix} 1-2b & 2u \\ 2u & 1-2b \end{pmatrix} \quad (82)$$

for the two bases $(b_1 \otimes f_1, f_1 \otimes b_1)$ and $(f_1 \otimes b_2, b_2 \otimes f_1)$,

$$\check{R}(u,b) = \frac{1}{1+2b+2u}\begin{pmatrix} 1+2b & 2u \\ 2u & 1+2b \end{pmatrix} \quad (83)$$

for the two bases $(b_1 \otimes f_2, f_2 \otimes b_1)$ and $(f_2 \otimes b_2, b_2 \otimes f_2)$,

(iii) a four-dimensional sub-matrix

$$\check{R}(u,b) = \frac{1}{d(u,b)}\begin{pmatrix} 1-4b^2+4u & n(u,b) & n(u,b) & 4u^2 \\ -n(u,b) & 1-4b^2 & -4u(1+u) & n(u,b) \\ -n(u,b) & -4u(1+u) & 1-4b^2 & n(u,b) \\ 4u^2 & -n(u,b) & -n(u,b) & 1-4b^2+4u \end{pmatrix} \quad (84)$$

† In this respect the results I obtain here seem to contradict the conclusions of [20].

where

$$n(u,b) = 2u(1+2b)^{1/2}(1-2b)^{1/2} \qquad d(u,b) = (1+2b+2u)(1-2b+2u) \tag{85}$$

for the basis $(b_1 \otimes b_2, f_1 \otimes f_2, f_2 \otimes f_1, b_2 \otimes b_1)$. The rational matrices have the block structure of their trigonometric parents, and they satisfy a Yang–Baxter equation with u as an additive spectral parameter. The parameter b appears explicitly in (77); it cannot be scaled away by rescaling the spectral parameter u.

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